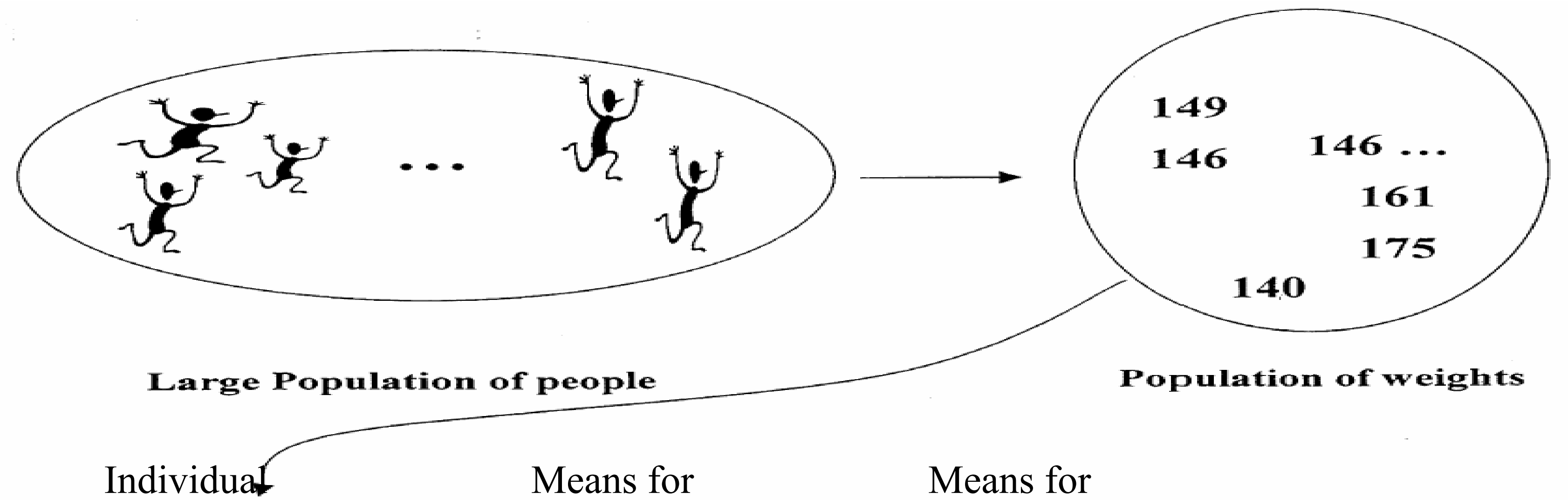
Module 13: Normal Distributions

This module focuses on the normal distribution and how to use it.

Reviewed 05 May 05/ MODULE 13

Sampling Distributions



Individual observations	Means for n = 5	Means for n = 20
149	153.0	151.6
146	146.4	151.3
. . .	. . .	. . .
μ = 150 lbs	μ = 150 lbs	μ = 150 lbs
$\sigma^2 = 100$lbs	$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} = 20 \text{ lbs}^2$	$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} = 5 \text{ lbs}^2$
σ = 10 lbs	$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = 4.47 \text{ lbs}$	$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = 2.23 \text{ lbs}$

Normal Distribution Density Function

The normal distribution is defined by the density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

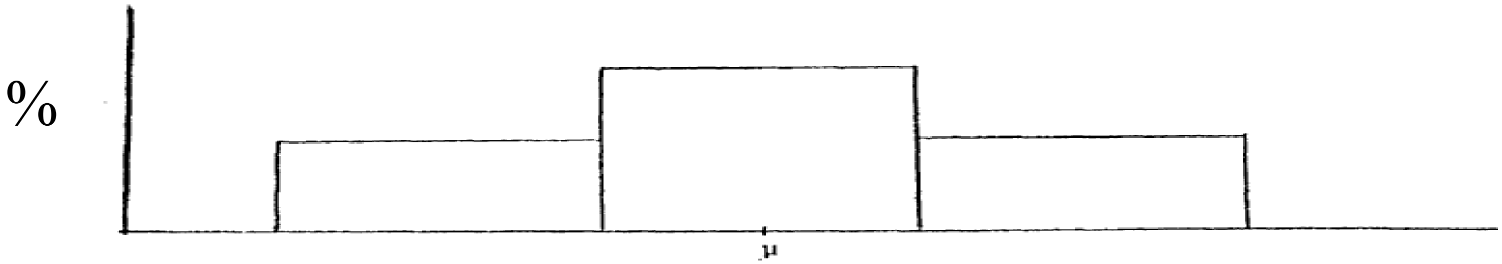
This function happens to be

Symmetrical,

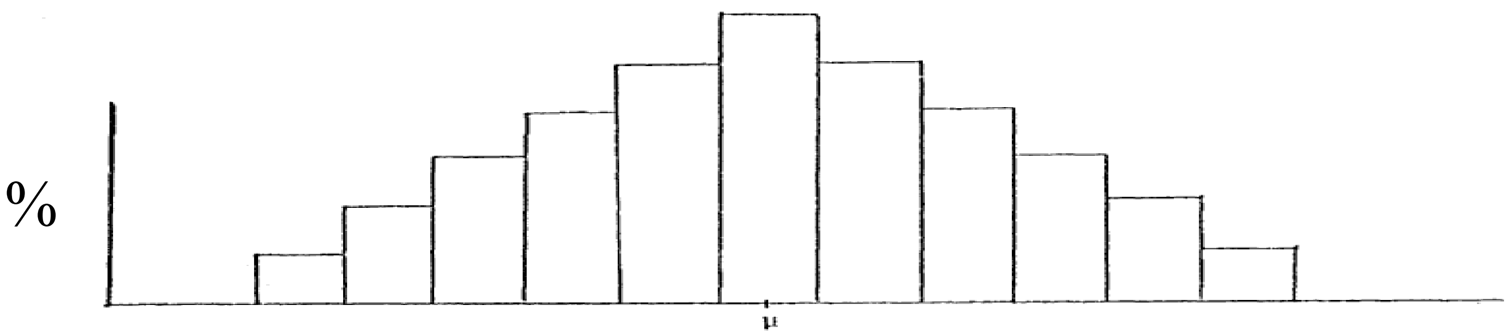
Bell-shaped,

and easy to use tables are available.

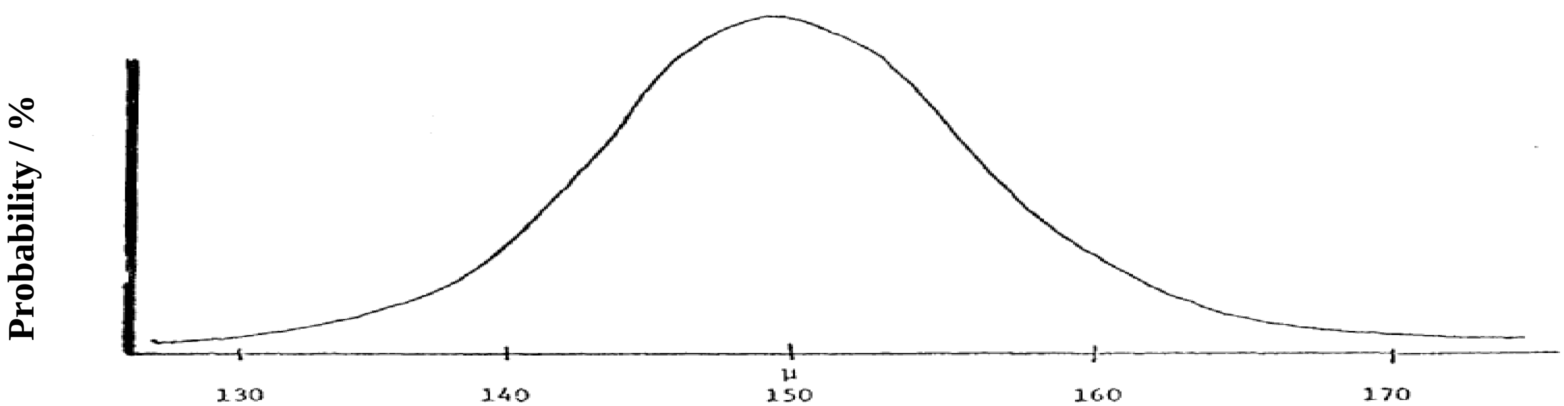
Normal Distribution



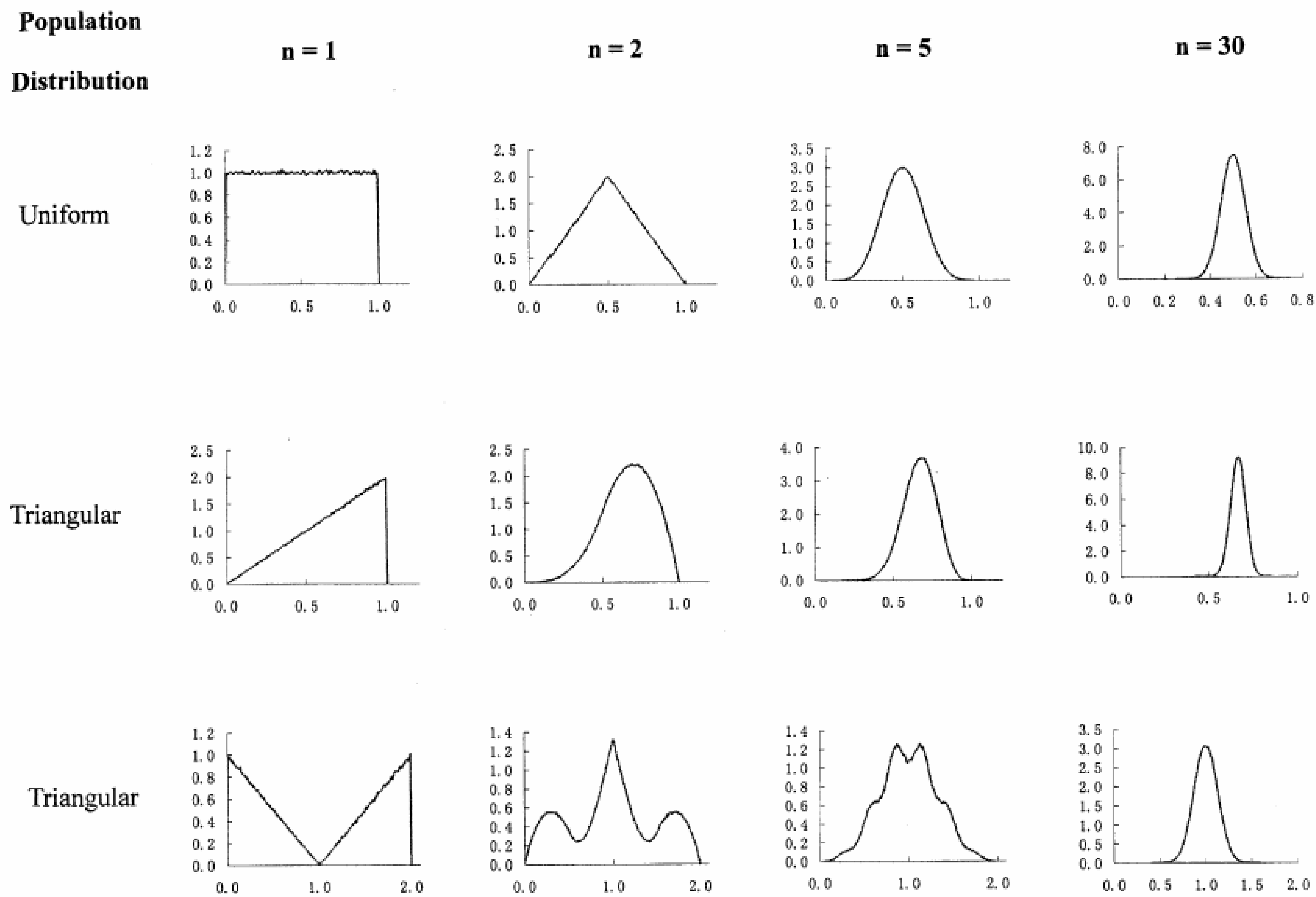
Based on eleven intervals, the histogram might be like:



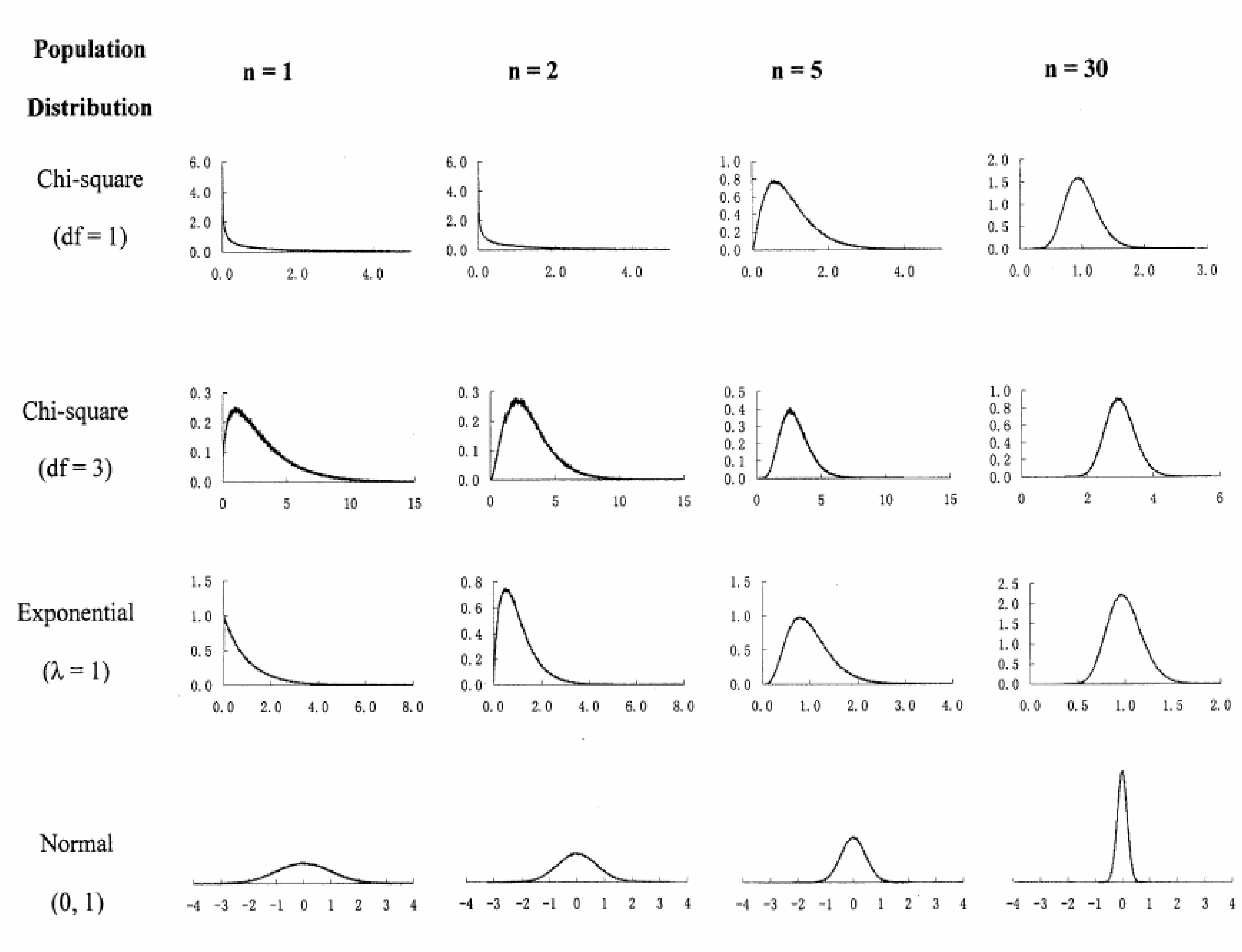
The histograms provide pictures of the distribution of the population of weights. If 1000 intervals were used, the picture might appear as:



Population Distributions



Population Distributions



Using the Normal Tables

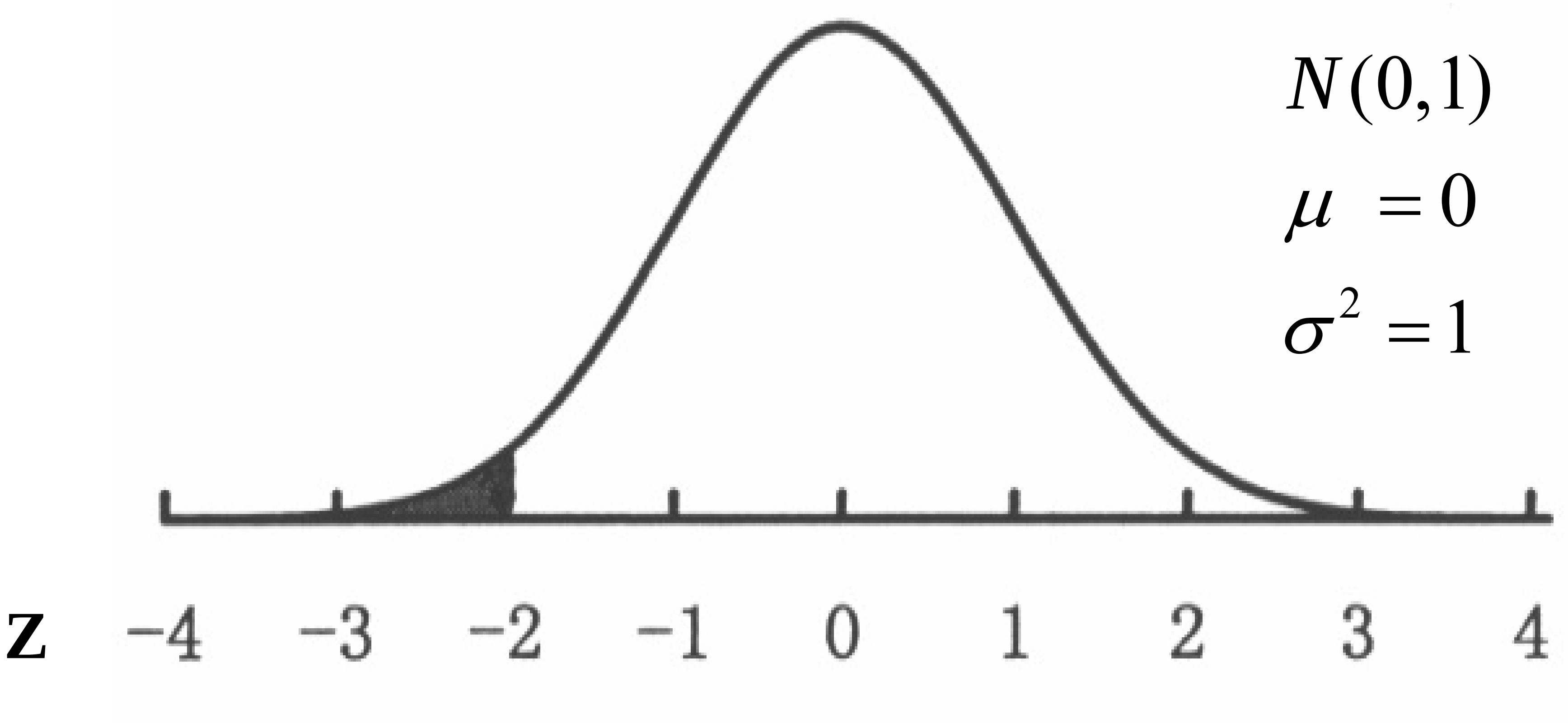
We can use the normal tables to obtain probabilities for measurements for which this frequency distribution is appropriate. For a reasonably complete set of probabilities, see TABLE MODULE 1: NORMAL TABLE.

This module provides most of the z-values and associated probabilities you are likely to use; however, it also provides instructions demonstrating how to calculate those not included directly in the table.

Normal Tables (contd.)

The table is a series of columns containing numbers for z and for P(z). The z represents the z-value for a normal distribution and P(z) represents the area under the normal curve to the left of that z-value for a normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$.

Using the Normal Tables

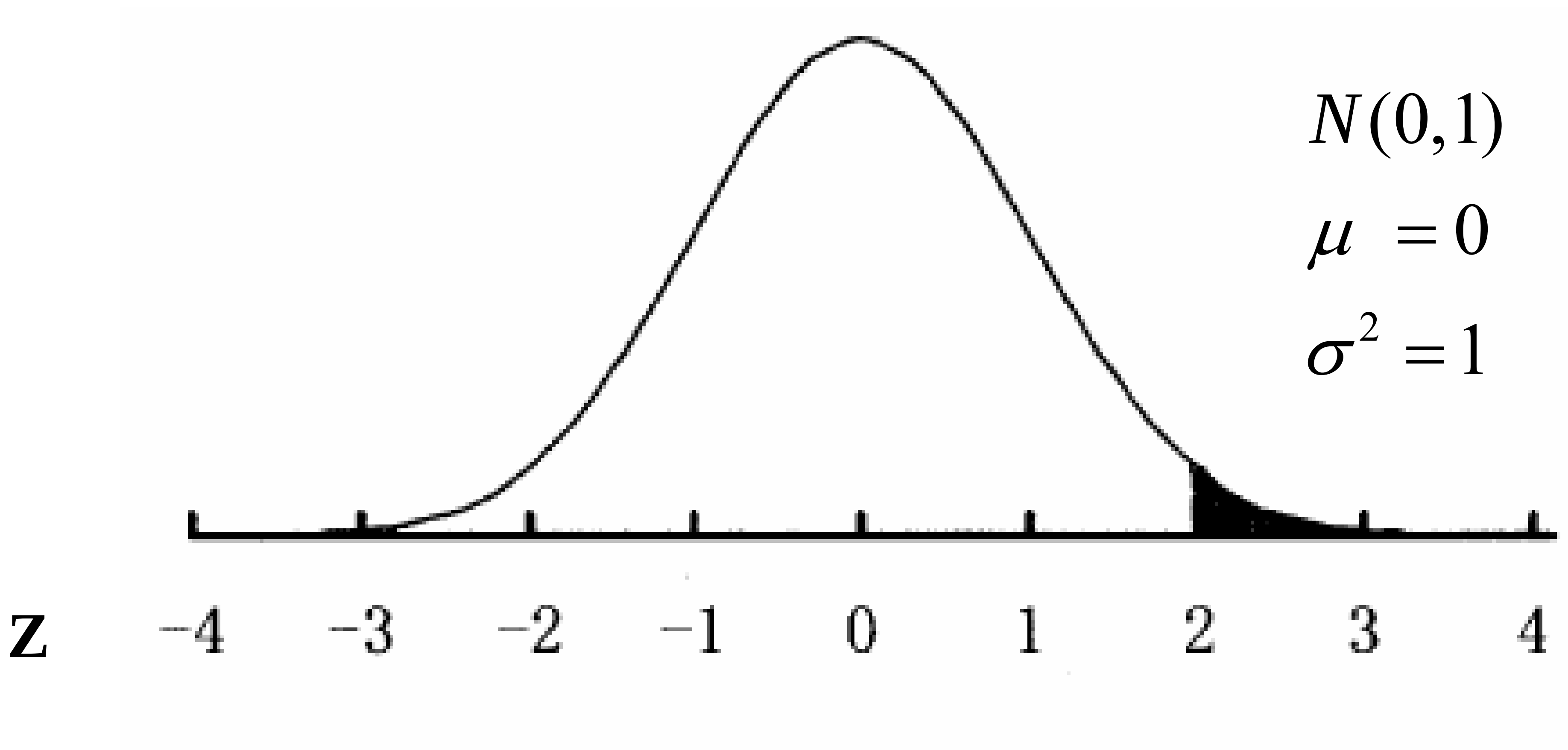


(1) Area Below z = -2; P(z < -2) = 0.0228

Using the Normal Tables

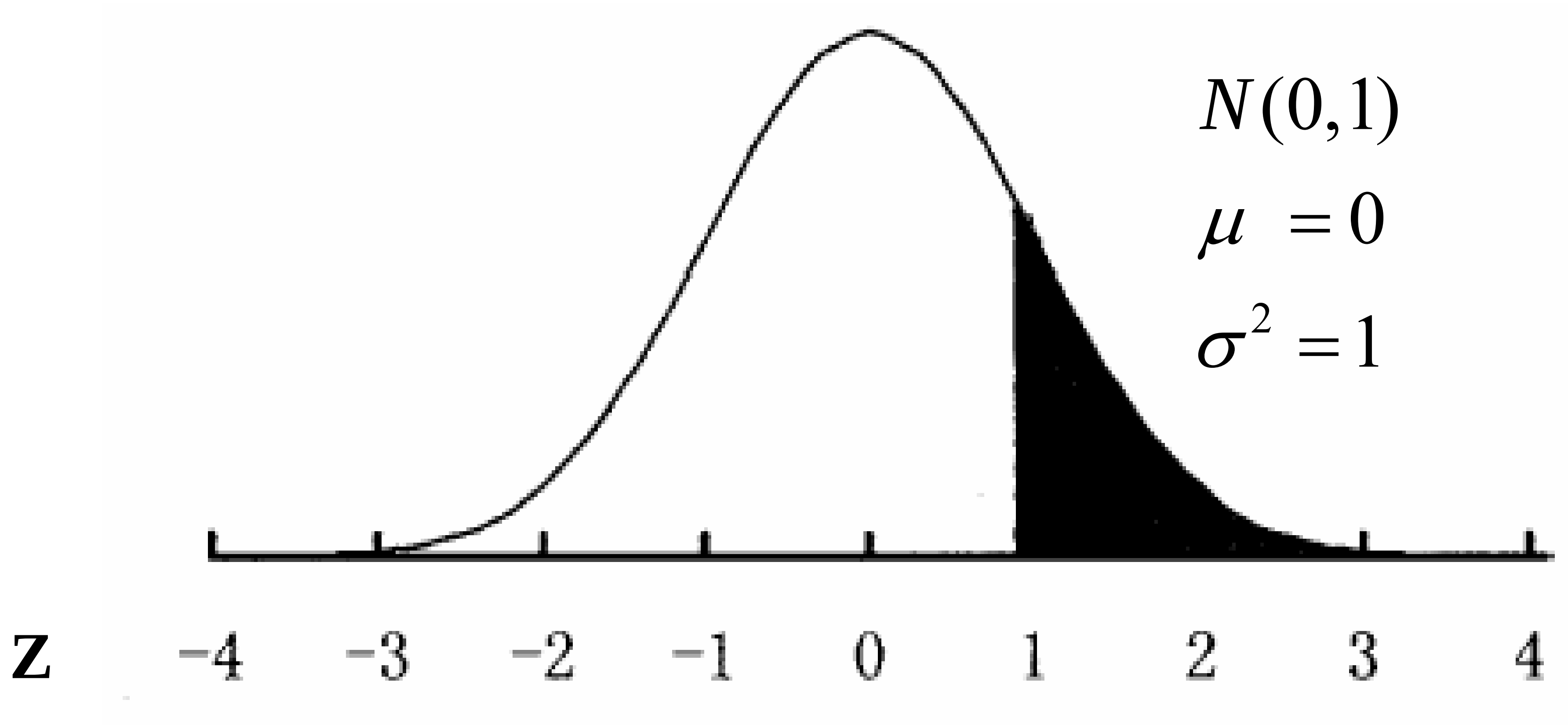
(2) Area Below z = -1; P(z < -1) = 0.1587

Using the Normal Tables



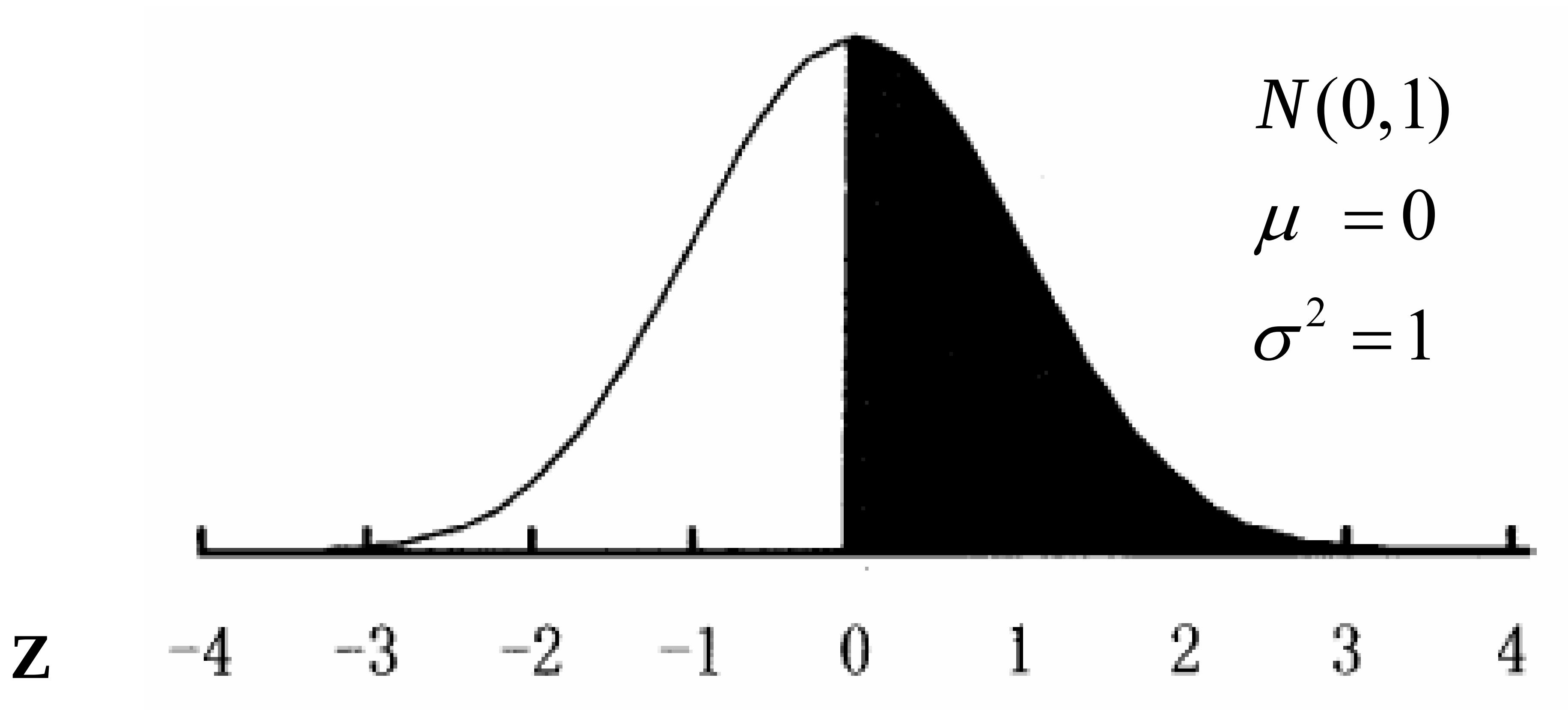
(1) Area Below z = +2; P(z > +2) = 0.0228

Using the Normal Tables



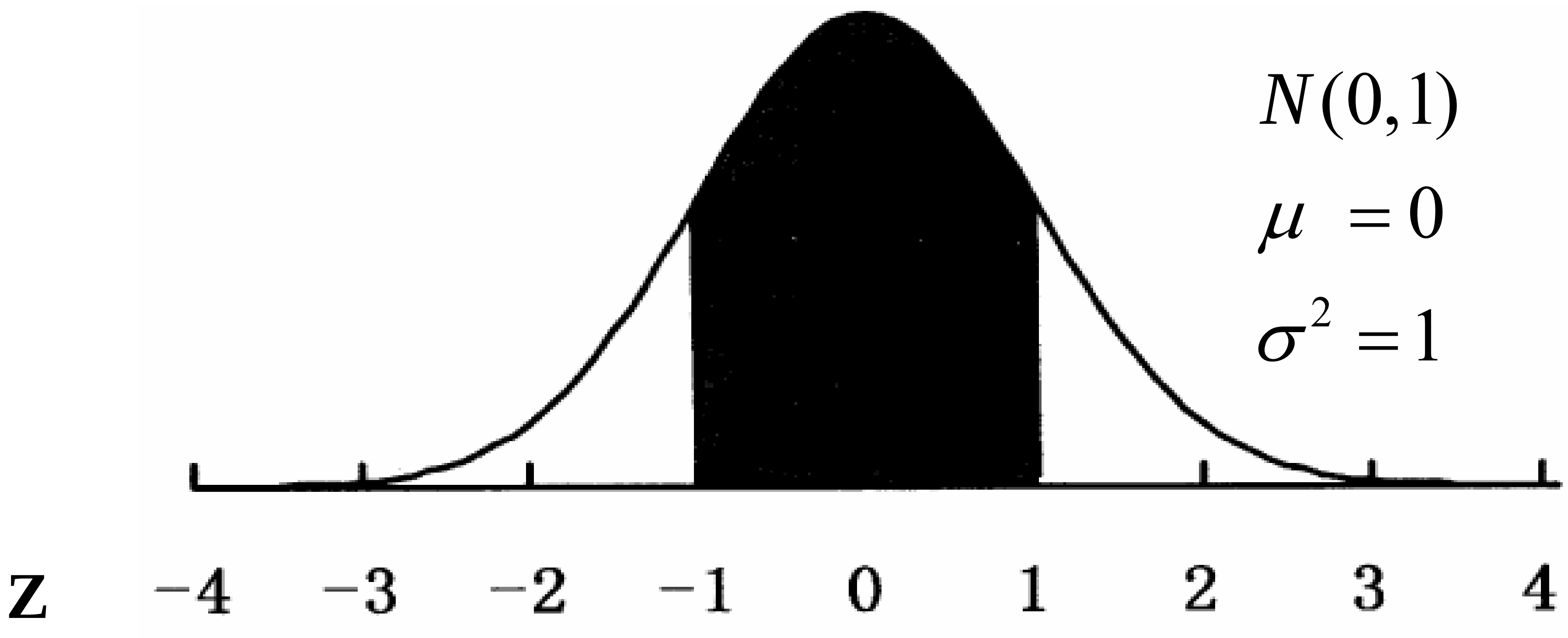
(2) Area Below z = +1; P(z > +1) = 0.1587

Using the Normal Tables



(3) Area Below z = 0; P(z > 0) = 0.5000

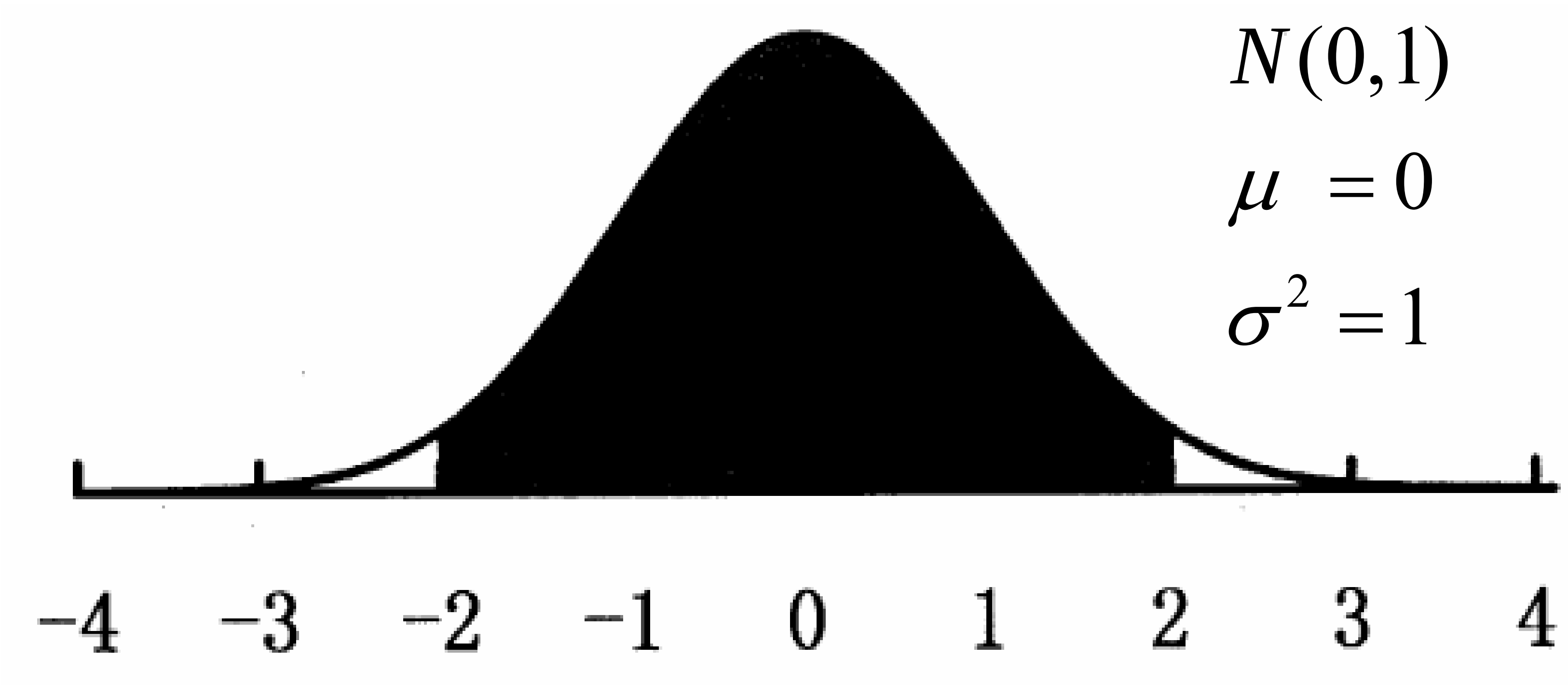
Calculating the Area Under the Normal Curve



(1) Area between -1, +1; P(-1 < z < +1)

up to z = +1:	**.8413**
up to z = -1 :	**.1587**
	.6826

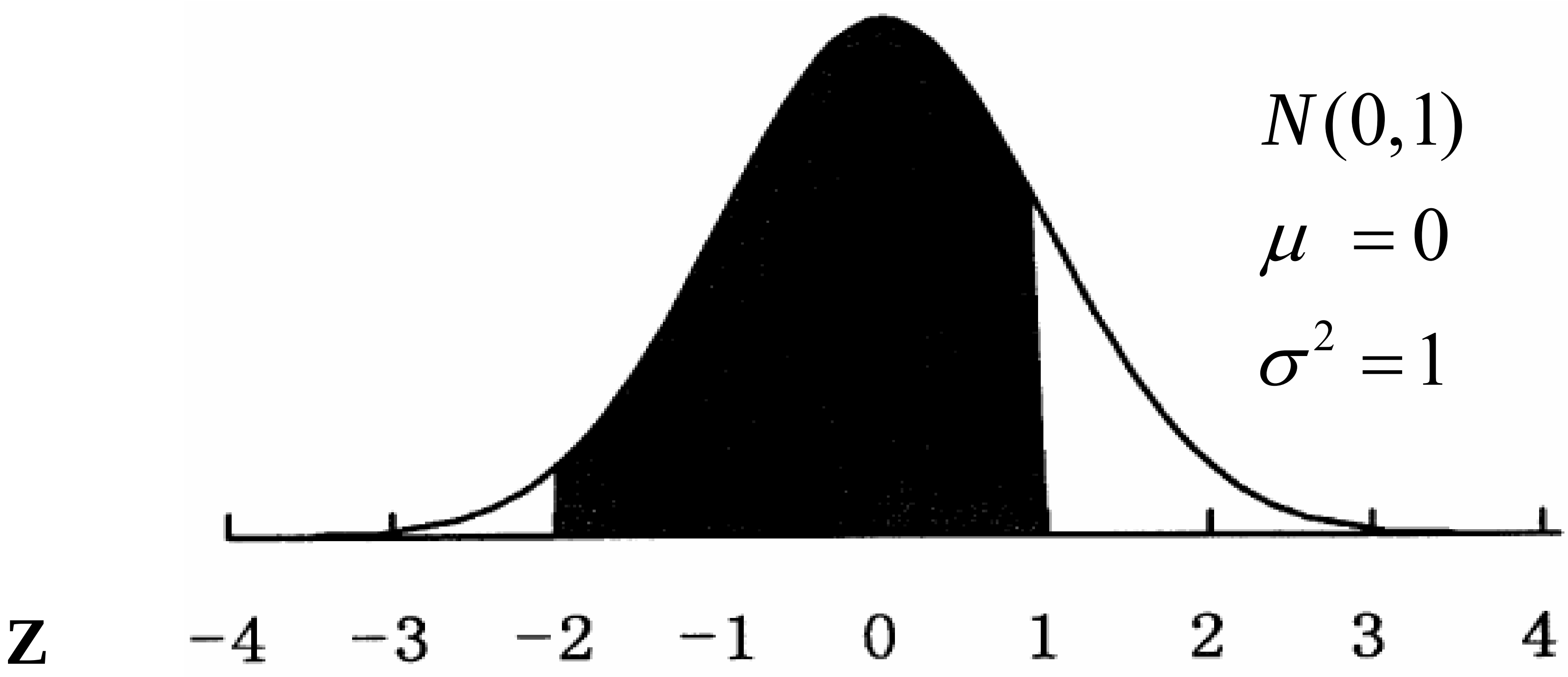
Calculating the Area Under the Normal Curve



(2) Area between -2, +2; $P(-2 < z < +2)$

up to z = +2:	**.9772**
up to z = -2 :	**.0228**
	.9544

Calculating the Area Under the Normal Curve



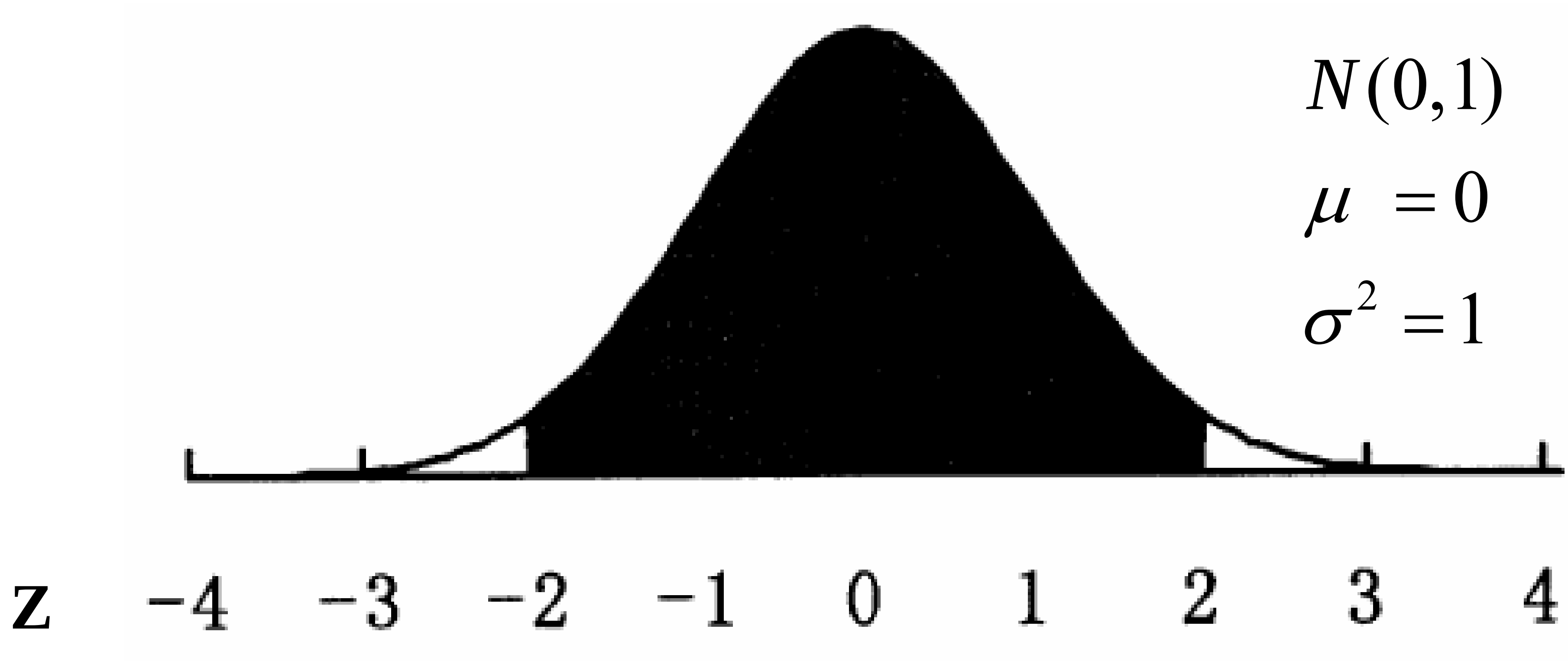
(3) Area between -2, +1; $P(-2 < z < +1)$

up to z = +1: .8413

up to z = -2 : .0228

.8185

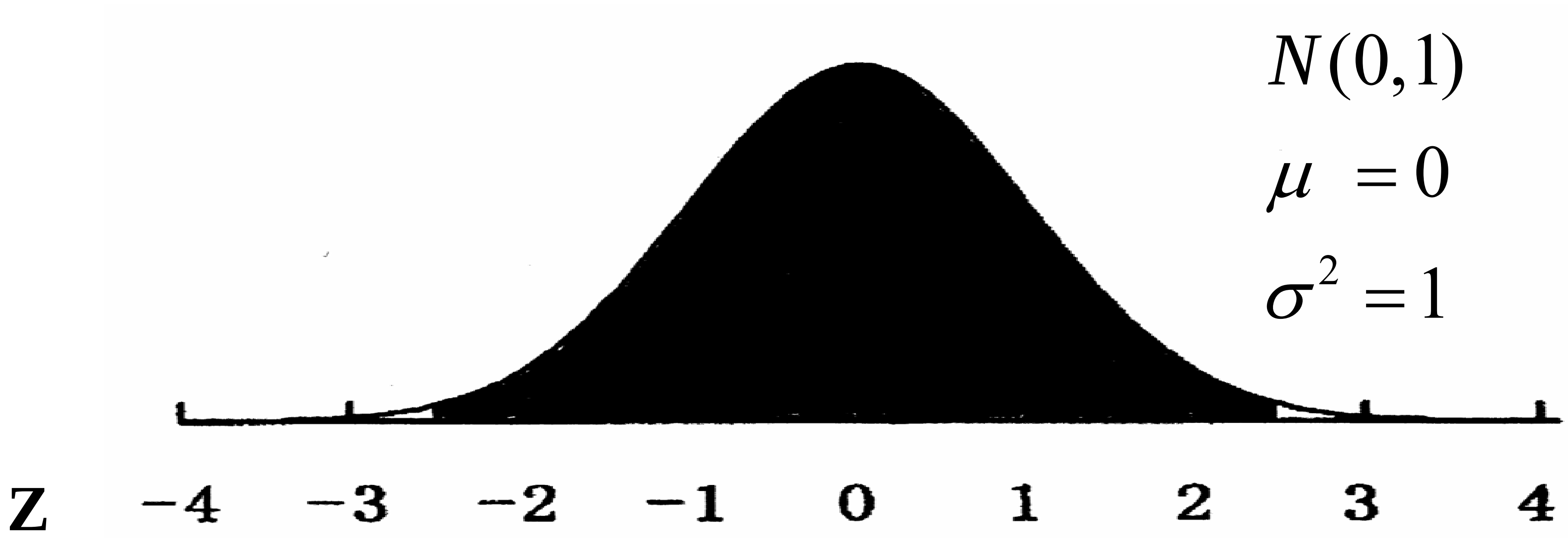
Standard Normal Distribution



(1) Values of z that bracket middle 95%

-1.96 to +1.96

Standard Normal Distribution



(1) Values of z that bracket middle 99%

-2.576 to +2.576

Calculating z-values

If $X \sim N(\mu_x, \sigma_x)$

and $Z \sim N(0,1)$

i.e. $\mu_z = 0$ and $\sigma_z^2 = 1$

then the corresponding z value for x is given as

$$z = \frac{x - \mu_x}{\sigma_x}$$

Calculating z-values

$Z \sim N(0,1)$

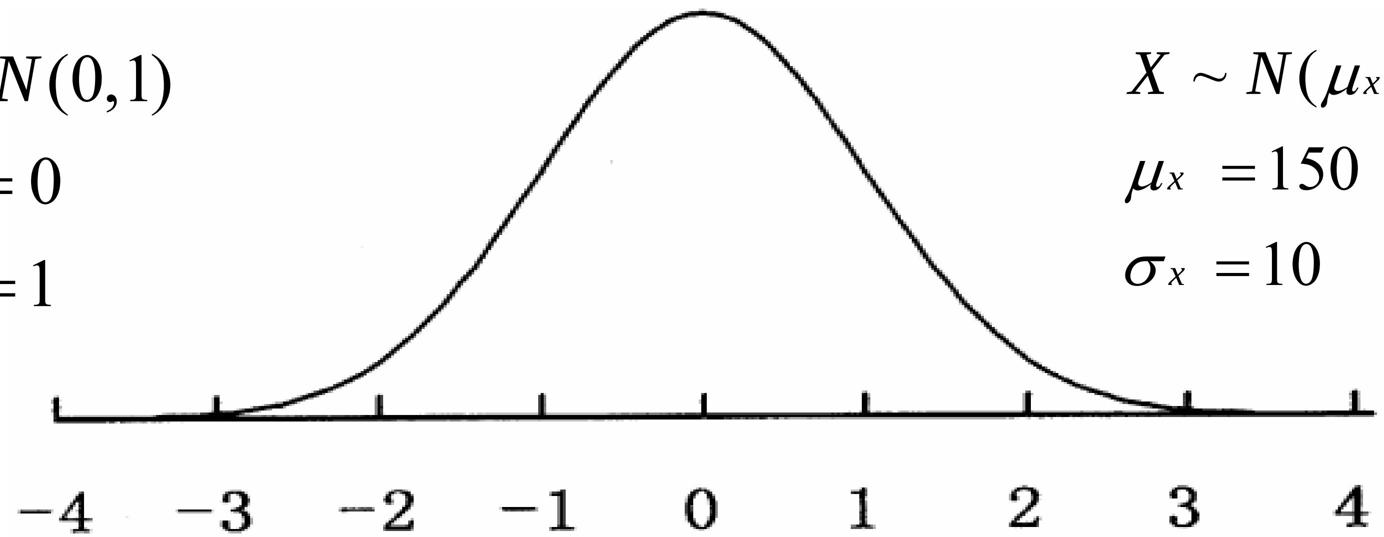
$\mu_Z = 0$

$\sigma_Z^{\,2} = 1$

$X \sim N(\mu_X, \sigma_X)$

$\mu_X = 150$

$\sigma_X = 10$



Z

Weight	110	120	130	140	150	160	170	180	190
		$\mu_x - 3\sigma_x$	$\mu_x - 2\sigma_x$	$\mu_x - 1\sigma_x$	μ_x	$\mu_x + 1\sigma_x$	$\mu_x + 2\sigma_x$	$\mu_x + 3\sigma_x$	

$z = \dfrac{x - \mu_x}{\sigma_x}$; if $X \sim \text{N}(150,10)$ *i.e.* $\mu_x = 150, \sigma_x = 10$

when $x = 150$; $z = \dfrac{150 - 150}{10} = 0$

when $x = 170$; $z = \dfrac{170 - 150}{10} = \dfrac{20}{10} = 2$

Some Questions

The following questions reference a normal distribution with a mean μ = 150 lbs, a variance σ^2 = 100 lbs^2, and a standard deviation σ = 10 lbs. Such a distribution is often indicated by the symbols $N(\mu,\sigma) = N(150, 10)$.

1. What is the likelihood that a randomly selected individual observation is within 5 lbs of the population mean μ = 150lbs?

2. What is the likelihood that a mean from a random sample of size n = 5 is within 5 lbs of μ = 150 lbs?

3. What is the likelihood that a mean from a random sample of size n = 20 is within 5 lbs of μ = 150 lbs?

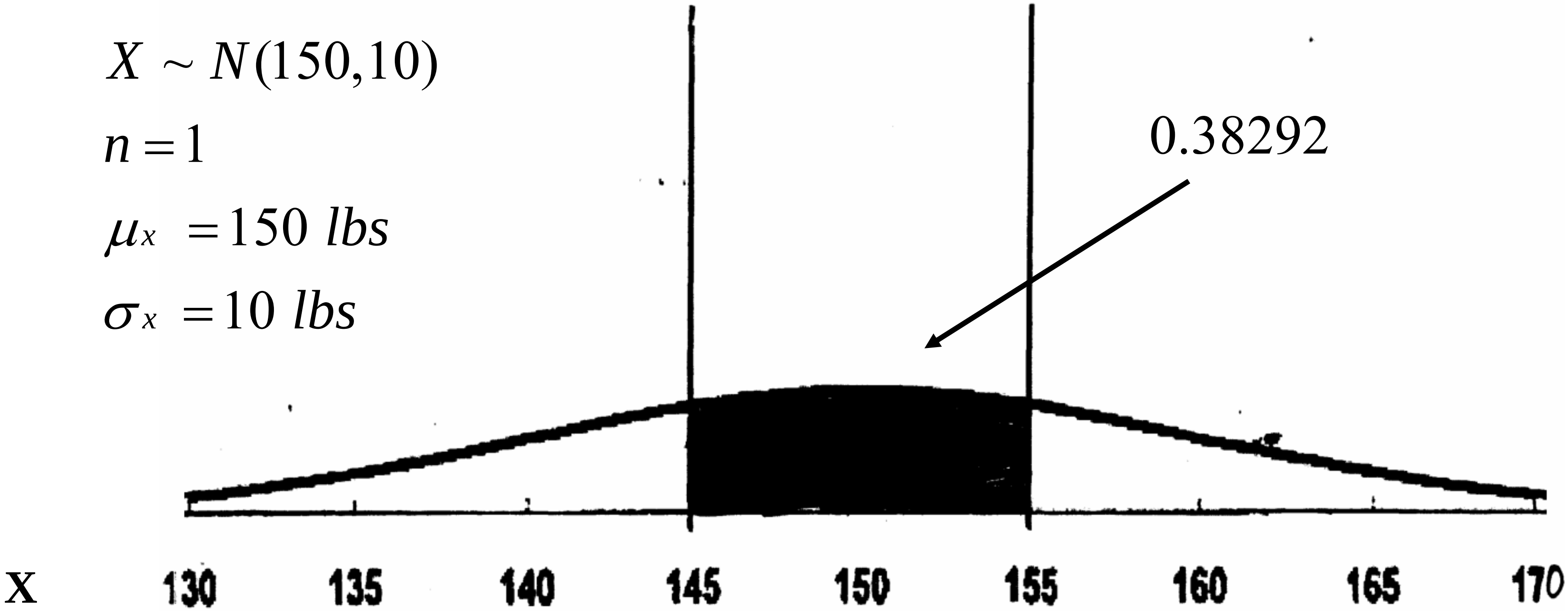
Solution to Question 1

$X \sim N(150,10)$

$n = 1$

$\mu_x = 150\ lbs$

$\sigma_x = 10\ lbs$



$z_{Upper} = \dfrac{x_{Upper} - \mu_x}{\sigma_x} = \dfrac{155-150}{10} = 0.5$, Area up to z $_{upper}$ = 0.69146

$z_{Lower} = \dfrac{x_{Lower} - \mu_x}{\sigma_x} = \dfrac{145-150}{10} = -0.5$, Area up to z $_{lower}$ = 0.30854

Area between z $_{upper}$ and z $_{lower}$ = 0.38292

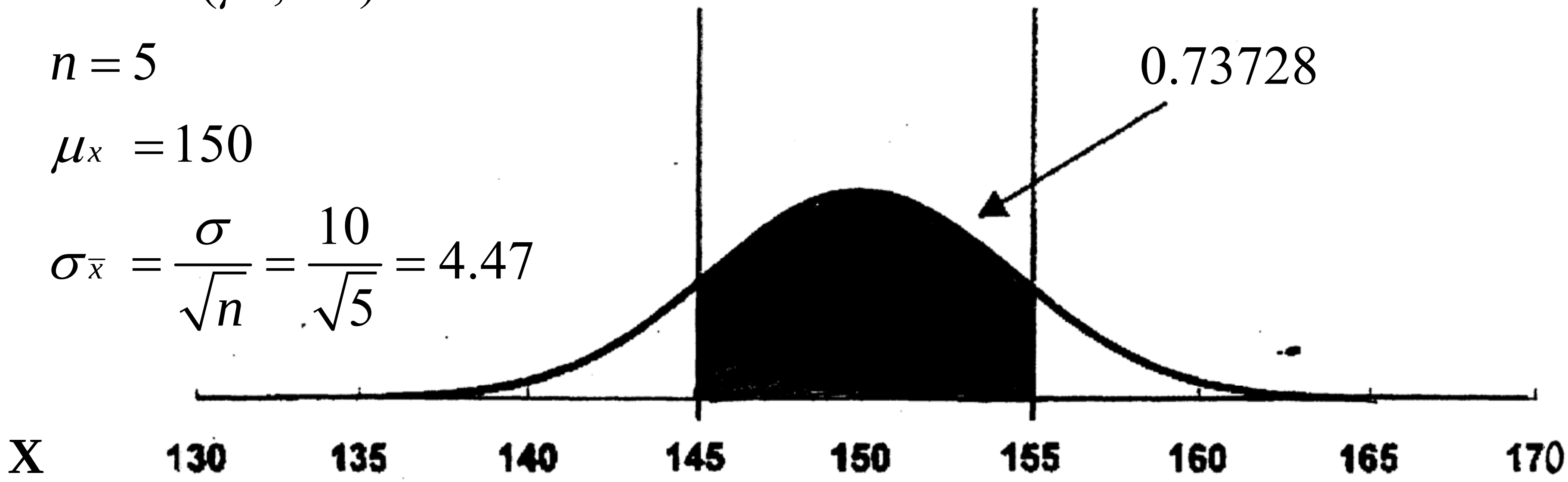
Solution to Question 2

$X \sim N(\mu_x, \sigma_x)$

$n = 5$

$\mu_x = 150$

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{5}} = 4.47$



$$z_{Upper} = \frac{x_{Upper} - \mu_x}{\sigma_x} = \frac{155 - 150}{4.47} = 1.12$$, Area up to z upper = 0.86864

$$z_{Lower} = \frac{x_{Lower} - \mu_x}{\sigma_x} = \frac{145 - 150}{4.47} = -1.12$$, Area up to z lower = 0.13136

Area between z upper and z lower = 0.73728

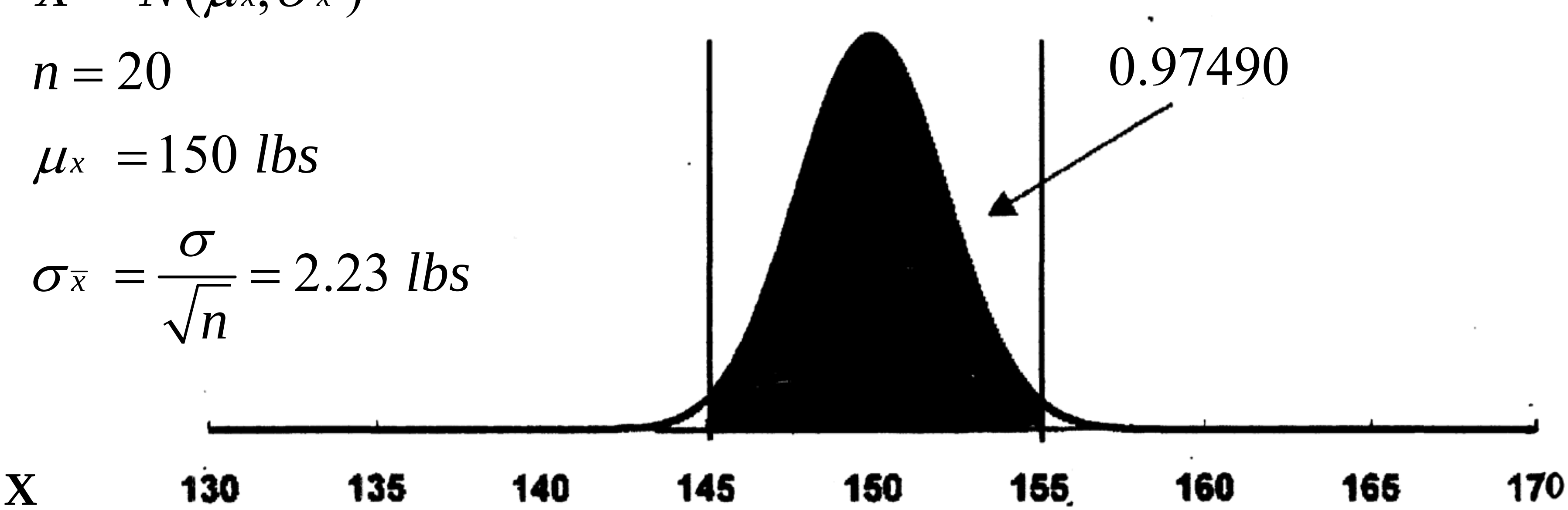
Solution to Question 3

$X \sim N(\mu_x, \sigma_x)$

$n = 20$

$\mu_x = 150\ lbs$

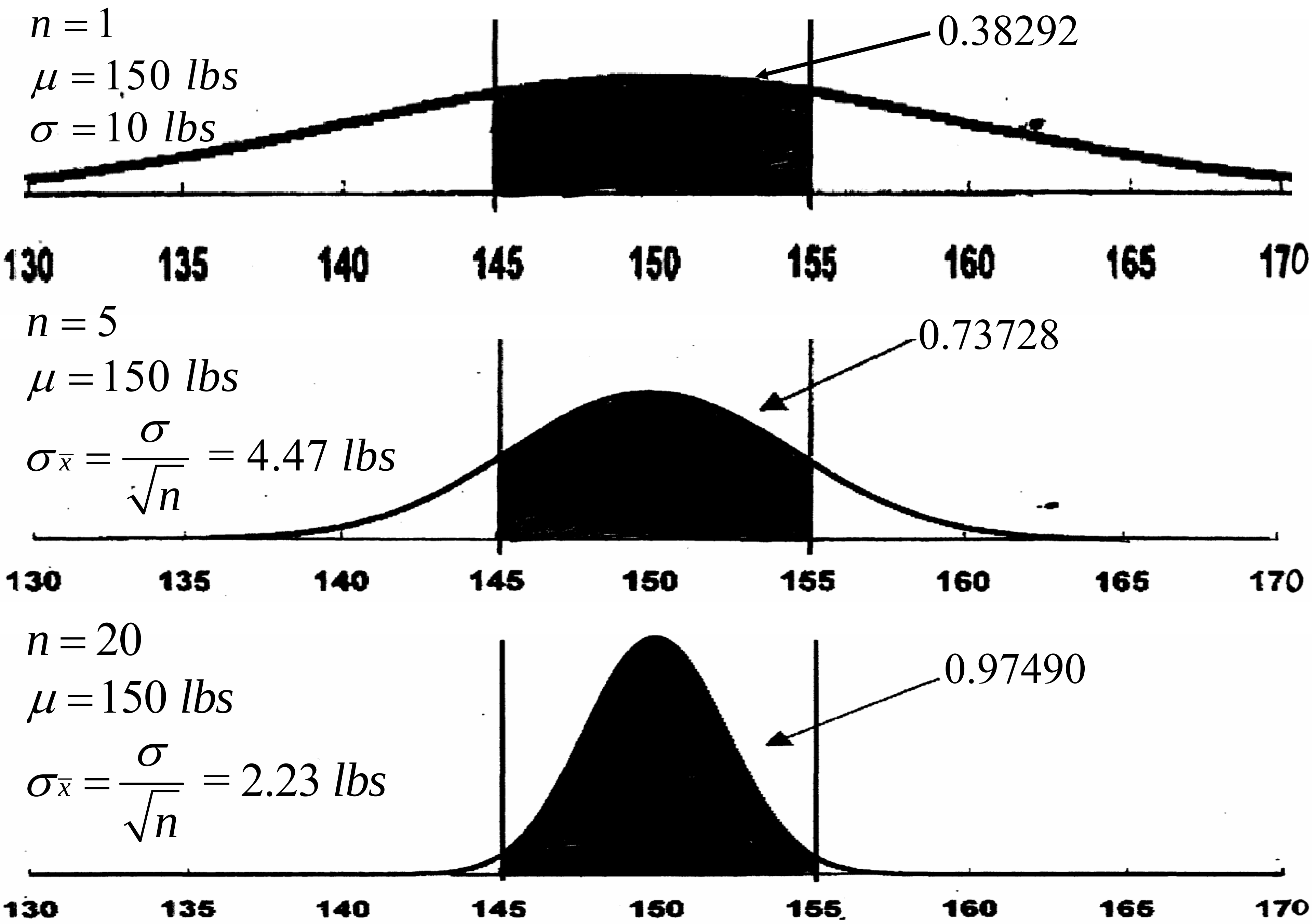
$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = 2.23\ lbs$



$z_{Upper} = \dfrac{x_{Upper} - \mu_x}{\sigma_x} = \dfrac{155 - 150}{2.23} = 2.24$, Area up to z $_{upper}$ = 0.98745

$z_{Lower} = \dfrac{x_{Lower} - \mu_x}{\sigma_x} = \dfrac{145 - 150}{2.23} = -2.24$, Area up to z $_{lower}$ = 0.01255

Area between z $_{upper}$ and z $_{lower}$ = 0.97490


n = 1
μ = 150 lbs
σ = 10 lbs
0.38292
130 135 140 145 150 155 160 165 170
n = 5
μ = 150 lbs
σx̄ = σ/√n = 4.47 lbs
0.73728
130 135 140 145 150 155 160 165 170
n = 20
μ = 150 lbs
σx̄ = σ/√n = 2.23 lbs
0.97490
130 135 140 145 150 155 160 165 170

Some More Questions

- When centered about $\mu = 150$ lbs, what proportion of the total distribution does an interval of length 10 lbs cover?
- How many standard deviations long must an interval be to cover the middle 95% of the distribution?
- From μ - (??) standard deviations to μ + (??) standard deviations covers (??) % of the distribution?

All these questions require that the value for μ be known and that it be placed in the center of these "intervals".